# BLOCK DEGENERACY AND CARTAN INVARIANTS FOR GRADED LIE ALGEBRAS OF CARTAN TYPE

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## 1. Introduction.

Let $L$ be a finite-dimensional Lie algebra over an algebraically closed field $F$ of characteristic $p \geq 5$. An element $x \in L$, $x \neq 0$, is an *absolute zero divisor* if $(\text{ad } x)^2 = 0$ [K1]. (In more recent terminology $x$ is sometimes referred to as a *sandwich element* [Z].) In the early 60's Kostrikin [K2] showed that these elements play a fundamental role in the structure theory of simple modular Lie algebras over prime characteristic. He called a Lie algebra containing an absolute zero divisor *strongly degenerate* and made the following bold conjecture which was proved 20 years later by Premet [Pr]:

**Kostrikin's Conjecture.** *Any finite dimensional simple Lie algebra over an algebraically closed field of prime characteristic is either classical or strongly degenerate.*

Block and Wilson [B-W] proved that all *restricted* simple Lie algebras are of either classical or of Cartan type. Recently, Strade and Wilson [St-W] have completed the classification for all simple Lie algebras (for $p > 7$) and have shown that these Lie algebras are also either of classical or Cartan type. The classical Lie algebras are the simple Lie algebras admitting a non-degenerate quotient trace form [Se2] and thus cannot be strongly degenerate. Therefore, Kostrikin's conjecture, along with these classification theorems, implies that, in the case of restricted simple Lie algebras, the Cartan type Lie algebras are exactly the ones with the strong degeneracy property.

In this paper we would like to illustrate a type of degeneracy which occurs in the block theory for Lie algebras of Cartan type. Recall if $L$ is a classical Lie algebra then $L$ has an simple module which is also projective [Hu3]. One often calls this the Steinberg module. It follows that $U(L)$, the restricted universal enveloping algebra [Ja], has at least two blocks or equivalently at least two central primitive idempotents. Our first objective will be to show this does not happen for Lie algebras of Cartan type, that is, if $L$ is a Lie algebra of Cartan type, then $U(L)$ has precisely one block (ie. no non-trivial central idempotents). A restricted Lie algebra will be of *one block type* if its restricted universal enveloping algebra has one block. In contrast to the complex semisimple Lie algebras, where the center of the universal enveloping algebra has no non-trivial nil ideals by the

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Harish-Chandra homomorphism theorem [Hu2], the center of the restricted universal enveloping algebra for a Lie algebra of Cartan type will be isomorphic to a finite-dimensional local algebra. Let $L = \oplus_{i=-n}^{m} L_i$ be a graded restricted Lie algebra. Later we will show how to compute the Cartan invariants of $U(L)$ relative to the Cartan invariants of $U(L_0)$. In particular for graded Lie algebras of Cartan type the problem of computing Cartan invariants will reduce to determining the Cartan invariants for the "classical" $L_0$ component. The authors would like to express their gratitude to Professor George B. Seligman for his help and assistance throughout this work.

## 2. Restricted Lie algebras with Triangular Decompositions.

Let $L$ be a restricted Lie algebra which admits a *triangular decomposition relative to a maximal torus* $T$:

$$L = N_L^- \oplus T \oplus N_L^+$$

where $N_L^-$ and $N_L^+$ are p-nilpotent restricted subalgebras. Set $B_L^{\pm} = T \oplus N_L^{\pm}$. We say that this decomposition for $L$ is *long* if

$$\dim_F N_L^- < \dim_F N_L^+.$$

By the standard arguments [Hu2] the simple modules for $L$ are in one-to-one correspondence with weights on the torus. Since we will be only considering restricted representations (ie. representations for $U(L)$) the simple modules will be parametrized by the set $\hat{T}$ of "restricted weights" [Se1]. Given such an $L$ we will provide theorems which give sufficient conditions to insure that $U(L)$ has precisely one block. In the next section we will show that the graded Lie algebras of Cartan type satisfy these conditions. We begin with a lemma which characterizes modules for certain subalgebras which will be contained in these Lie algebras of Cartan type. For $M$, a $U(L)$ module, let $[M]$ denote the formal sum of composition factors in the Grothendieck ring of $U(L)$.

**Lemma 2.1.** *Let $Q$ be a restricted Lie algebra with a triangular decomposition relative to a maximal torus $T$:*

$$Q = N_Q^- \oplus T \oplus N_Q^+.$$

*Assume the following:*

(1) *$N_Q^- \oplus N_Q^+$ is a p-ideal in $Q$ and*
(2) *$N_Q^+$ contains $\dim_F T$ linearly independent weight vectors having linearly independent weights in $\hat{T}$.*

*For each $\lambda \in \hat{T}$ set $V(\lambda) = \bar{D}_\lambda \otimes_{U(B_Q^-)} U(Q)$ where $\bar{D}_\lambda$ is a one dimensional simple $U(B_Q^-)$ module corresponding to weight $\lambda$. Then $[V(\lambda)]$ is independent of $\lambda$ and*

$$[V(\lambda)] = \sum_{\mu \in \hat{T}} p^{\beta} [D_\mu]$$

*where* $\beta = dim_F N_Q^+ - dim_F T$*, and* $D_\mu$ *is the one dimensional simple* $U(Q)$ *module of weight* $\mu$*.*

*Proof.* By assumption (1), $N_Q^- \oplus N_Q^+$ is a p-ideal in $Q$ which is p-nilpotent. It follows that $Q = T \oplus rad_p(Q)$ where $rad_p(Q) = N_Q^- \oplus N_Q^+$ is the largest p-nilpotent ideal of $Q$. This implies $Q$ is completely solvable. Hence, all the restricted representations for $Q$ are one dimensional [S-F] and we can let $\{D_\mu : \ \mu \in \hat{T}\}$ represent the set of non-isomorphic simple $U(Q)$ modules. In the case of completely solvable Lie algebras the composition factors for a module can be obtained by finding its weight space decomposition. From assumption (2) as $U(T)$ module $U(N_Q^+)$ must have all possible weights occurring with the same multiplicity, $p^\beta$, where $\beta = dim_F N_Q^+ - dim_F T$. Hence, all possible one-dimensional irreducible modules must occur with multiplicity $p^k$ in $V(\lambda)$ (since the weight space decomposition is obtained by looking at $U(N_Q^+)$ as $U(T)$ module and "shifting" by $\lambda$).□

**Theorem 2.2.** *Let* $L$ *be a restricted Lie algebra which admits a long triangular decomposition:*

$$L = N_L^- \oplus T \oplus N_L^+.$$

*Set* $V^\pm(\sigma) = D_\sigma \otimes_{U(B_L^\pm)} U(L)$ *for all* $\sigma \in \hat{T}$*. Assume the following;*

(1) *$L$ has a restricted subalgebra $Q$, satisfying the assumptions in Lemma 2.1.*
(2) *$B_Q^- = B_L^-$ and*
(3) *$N_Q^- = N_L^-$ has at least $\dim_F T$ linearly independent weight vectors having linearly independent weights in $\hat{T}$.*

*Then*

$$[V^-(\lambda)] = \sum_{\mu \in \hat{T}} p^\beta [V^+(\mu)] \qquad (\dagger)$$

*where* $\beta = dim_F N_L^+ - dim_F N_L^- - dim_F T$*.*

*Proof.* First we will show $[V^-(\lambda)]$ is independent of $\lambda \in \hat{T}$. By the previous lemma we have:

$$\begin{aligned} [V^-(\lambda)] &= [D_\lambda \otimes_{U(B_L^-)} U(L)] \\ &= [[D_\lambda \otimes_{U(B_L^-)} U(Q)] \otimes_{U(Q)} U(L)] \\ &= \sum_{\mu \in \hat{T}} p^k [D_\mu \otimes_{U(Q)} U(L)] \end{aligned}$$

where $k = dim_F Q - dim_F N_L^-$. Therefore, $[V^-(\lambda)]$ is independent of $\lambda$.

Now using assumption (3) and the fact that the assumption holds when we replace $N_Q^-$ with $N_Q^+$ (because of the hypotheses on $Q$) it follows that

$$[D_\lambda \otimes_{U(T)} U(L)] = [D_\lambda \otimes_{U(T)} U(B_L^\pm) \otimes_{U(B_L^\pm)} U(L)] = \sum_{\mu \in \hat{T}} p^{k_\pm}[D_\mu \otimes_{U(B_L^\pm)} U(L)]$$

where $k_\pm = dim_F N_L^\pm$. Since $dim_F N_L^+ > dim_F N_L^-$ we have

$$\sum_{\mu \in \hat{T}} [D_\mu \otimes_{U(B_L^-)} U(L)] = \sum_{\mu \in \hat{T}} p^{k_+ - k_-}[D_\mu \otimes_{U(B_L^+)} U(L)].$$

But, $[V^-(\lambda)]$ is independent of $\lambda \in \hat{T}$. Therefore, for all $\lambda \in \hat{T}$

$$p^{\dim_F T}[V^-(\lambda)] = \sum_{\mu \in \hat{T}} p^{k_+ - k_-}[V^+(\mu)]$$

and

$$[V(\lambda)] = \sum_{\mu \in \hat{T}} p^{k_+ - k_- - dim_F T}[V^+(\mu)].\square$$

There exist restricted Lie algebras admitting long decompositions which do not satisfy the conditions of Lemma 2.1. However, the following is a theorem which provides conditions in which (†) (in Theorem 2.2) still holds.

**Theorem 2.3 ([N], Thm 3.1.2.).** *Let $L$ be a restricted Lie algebra with a long triangular decomposition:*

$$L = N_L^- \oplus T \oplus N_L^+.$$

*Assume that there exists a subalgebra $Q$ such that*

(1) $B_L^- \subset Q$,
(2) *$Q$ is a classical Lie algebra and*
(3) *a vector space complementary to $Q$, in $L$, contains $dim_F T$ weight vectors having linearly independent weights in $\hat{T}$.*

*Then*

$$[V^-(\lambda)] = \sum_{\mu \in \hat{T}} p^\beta [V^+(\mu)]$$

*where $\beta = dim_F N_L^+ - dim_F N_L^- - dim_F T$.*

**Corollary 2.4.** *If $L$ is a restricted Lie algebra satisfying the assumptions in Theorem 2.2 or Theorem 2.3 then $U(L)$ has one block.*

*Proof.* Let $\mathcal{P}(\mu)$ be a projective indecomposable module for $U(L)$ with simple head $\mathcal{L}(\mu)$. Moreover, let $V^-(\mu)$ be the Verma module with simple head $\mathcal{L}(\mu)$. Since $\mathcal{P}(\mu)$ is projective and $V^-(\mu)$ has a unique maximal submodule there exists a surjective homomorphism $\mathcal{P}(\mu) \twoheadrightarrow V^-(\mu)$. Therefore, all the composition factors of $V^-(\mu)$ must be composition factors of $\mathcal{P}(\mu)$. According to Theorem 2.2. or Theorem 2.3. $V^-(\mu)$ has every irreducible occuring as a factor, hence the same must be true for $\mathcal{P}(\mu)$.□

From standard theorems in [Cu-R] we can deduce information about the structure of the center of the u-algebra when there is only one central idempotent.

**Corollary 2.5.** *Let $L$ be a restricted Lie algebra satisfying the assumptions in either Theorem 2.2 or Theorem 2.3. Then the center of $U(L)$ is isomorphic to a finite-dimensional local algebra.*

## 3. The Lie algebras of Cartan type are of one block type.

Throughout this section we will use the conventions and notation in [S-F]. The restricted universal enveloping algebras for the Lie algebras of types W and K were shown in [N] to have one block by using Theorem 2.3. We will extend these results to all the graded restricted Lie algebras of Cartan type by finding a subalgebra Q satisfying the conditions of Theorem 2.2.

### Type W (W(n,1) for n>1 and p>3).

For $W(1,1)$ there exists no subalgebra Q satisfying the hypotheses of Theorem 2.2. In this case use Theorem 2.3. Let $L = W(n,1)$, $n > 1$. Then $L = L_{-1} \oplus L_0 \oplus L_1 \oplus ...L_t$ with $L_i = \langle x^a D_j : \mid a \mid = i+1\rangle$, $\mid a \mid = \sum_{i=1}^n a_i$. From the gradation one obtains a long triangular decomposition $L = N^- \oplus T \oplus N^+$ with

$$\begin{aligned} N^- &= \oplus_{i<0} L_i \oplus \langle x^{\epsilon_i} D_j : \ i > j\rangle \\ T &= \langle x^{\epsilon_j} D_j : \ j = 1, 2, ...n\rangle \\ N^+ &= \oplus_{i>0} L_i \oplus \langle x^{\epsilon_i} D_j : \ i < j\rangle. \end{aligned}$$

Set $Q = N_Q^- \oplus T \oplus N_Q^+$ where $N_Q^- = N^-$ and $N_Q^+ = \langle x^a D_1 : \ a_1 = 0, \ \mid a \mid \geq 2\rangle$.

In order to prove that $Q$ is a subalgebra satisfying (1) in lemma 2.1 it suffices to prove that $N_Q^- \oplus N_Q^+$ is closed under brackets. It is clear that $[N_Q^+, N_Q^+] \subset N_Q^+$. Moreover,

$$[D_j, x^a D_1] = -x^{a-\epsilon_j} D_1.$$

This is in $N_Q^+$ if $\mid a \mid \geq 3$. If $\mid a \mid = 2$ then it is in $\langle x^{\epsilon_i} D_j :\ i > j \rangle$. For $i > j$ we have

$$[x^{\epsilon_i} D_j, x^a D_1] = \delta_{i,1} x^a D_j - x^{\epsilon_i} x^{a-\epsilon_j} D_1 = -x^{\epsilon_i} x^{a-\epsilon_j} D_1$$

which is still in $N_S^+$.

For condition (2) of the lemma 2.1 observe that the element $x^{3\epsilon_2} D_1$ has weight $(-1, 3, 0, ...0)$ and for $2 \leq i < n$ the element $x^{2\epsilon_i} D_1$ has weight $(-1, 0, 0, ...0) + 2\epsilon_i$ with respect to the maximal torus $T$. It is easy to see that these weights are linearly independent.

**Type S (S(n,1) for n>2 and p>3).**

Let $L = S(n, 1)$. For each pair $i, j \in \{1, 2, ...n\}$ define $D_{ij} : A(n, 1) \to W(n, 1)$ by

$$D_{ij}(f) = D_j(f) D_i - D_i(f) D_j.$$

Then $L = \langle D_{ij}(f) :\ \ f \in A(n, 1),\ 1 \leq i < j \leq n \rangle$. The maps above induce a grading: $L = L_{-1} \oplus L_0 \oplus L_1 \oplus ... L_t$ with $L_k = \langle D_{ij}(x^a) : \mid a \mid = k - 2 \rangle$. Set $Q = N_Q^- \oplus T \oplus N_Q^+$ with

$$\begin{aligned}
&N_Q^- = \langle D_j :\ 1 \leq j \leq n \rangle \oplus \langle x^{\epsilon_i} D_j :\ i > j \rangle \\
&T = \langle x^{\epsilon_j} D_j - x^{\epsilon_{j+1}} D_{j+1} :\ j = 1, 2, ...n - 1 \rangle \\
&N_Q^+ = \langle D_{1j}(x^a) = x^{a-\epsilon_j} D_1 :\ a_1 = 0,\ \mid a \mid \geq 3 \rangle.
\end{aligned}$$

First observe that $[N_Q^+, N_Q^+] = 0$. Moreover, we have

$$\begin{aligned}
&[x^{\epsilon_i} D_j, x^{a-\epsilon_k} D_1] = -x^{\epsilon_i} x^{a-\epsilon_k-\epsilon_j} D_1 and \\
&[D_j, x^{a-\epsilon_k} D_1] = -x^{a-\epsilon_k-\epsilon_j} D_1.
\end{aligned}$$

The last element is in $N_Q^+$ if $\mid a \mid \geq 4$. If $\mid a \mid = 3$ then it is in $\langle x^{\epsilon_i} D_j :\ i > j \rangle$. Therefore, $N_Q^- \oplus N_Q^+$ is an ideal contained in the subalgebra $Q$ and (1) of Lemma 2.1 is satisfied.

Now set $\beta_i = (0, 1, ...1, \overset{i}{2}, ...2)$ for $2 \leq i \leq n$. Then for $2 \leq i \leq n - 1$ $D_{1i}(x^{\beta_i})$ has weight $(-2, 0, ...0, \overset{i}{-1}, 0, ...0)$ while $D_{1n}(x^{\beta_n})$ has weight $(-2, 0, ...0)$ with respect to $T$. Hence (2) is satisfied for S(n,1).

**Type H (H(2r,1) n=2r for n>1 p>3).**

Let $L = H(2r, 1)$. We have a map $D_H : A(2r, 1) \to H(2r, 1)$ given by

$$D_H(f) = \sum_{j=1}^{r} D_j(f) D_{j+r} - \sum_{j=r+1}^{2r} D_j(f) D_{j-r}$$

with $H(2r,1) = \langle D_H(x^a):\ 0 \le a_i \le p-1\rangle$. The map $D_H$ induces a grading on $L$: $L = L_{-1} \oplus L_0 \oplus L_1 \oplus ...L_t$ where $L_i = \langle D_H(x^a) : \mid a \mid = i-2\rangle$.

Now set $Q = N_Q^- \oplus T \oplus N_Q^+$ where

$$N_Q^- = \langle D_j:\ j = 1,2,...n\rangle \oplus \langle D_H(x^{\epsilon_i+\epsilon_j}):\ i,j \ge r\ or\ (i<r,\quad j \ge r\ and\ j-r<i)\rangle$$
$$T = \langle D_H(x^{\epsilon_j+\epsilon_{j+r}}):\ j = 1,2,...r\rangle$$
$$N_Q^+ = \langle D_H(x^a):\ a_i = 0\ for\ i \le r,\ \mid a \mid \ge 3\rangle$$

Once again one can easily check that $[N_Q^+, N_Q^+] = 0$ and $[N_Q^-, N_Q^+] \subset N_Q^+$. (If $i,j$ satisfy $i<r$, $j \ge r$ and $j-r<i$ then $[D_H(x^{\epsilon_i+\epsilon_j}), D_H(x^a)]$ is in the span of $D_H(x^{a+\epsilon_j-\epsilon_{i+r}})$). Therefore, $N_Q^- \oplus N_Q^+$ is an ideal in $Q$. Moreover, observe that $D_H(x^{3\epsilon_{i+r-1}})$ has weight $(0,...0,\overset{i}{-3},0,...0)$ so the conditions of Lemma 2.1 are satisfied.

**Type K (K(2r+1,1) n=2r+1 n>1 p>3).**

Let $L = K(2r+1,1)$, $n = 2r+1$. There exists a linear isomorphism $D_K : A(n,1) \to K(n,1)$ defined by $D_K(f) = \sum_{j=1}^n f_j D_j$ where

$$f_j = x^{\epsilon_j} D_n(f) - D_{j+r}(f)\ for\ \ 1 \le j \le r$$
$$f_j = x^{\epsilon_j} D_n(f) + D_{j-r}(f)\ for\ \ r+1 \le j \le 2r$$
$$f_n = 2f - \sum_{j=1}^{r} x^{\epsilon_j} f_{j+r} + \sum_{j=r+1}^{2r} x^{\epsilon_j} f_{j-r}$$

We obtain a grading $L = L_{-2} \oplus L_{-1} \oplus ...L_t$ by setting $L_i = \langle D_K(x^a) : \mid a \mid = i-2\rangle$ where $\mid a \mid = \sum_{i=1}^{2r} a_i + 2a_n$. Let $Q = N_Q^- \oplus T \oplus N_Q^+$ with

$$N_Q^- = \langle D_K(1)\rangle \oplus \langle D_K(x^{\epsilon_i}):\ 1 \le i \le 2r\rangle$$
$$\oplus \langle D_K(x^{\epsilon_i+\epsilon_j}):\ r<i,\ j \le 2r\ or\ (i \le r,\ r<j \le 2r,\quad j-r<i)\rangle$$
$$T = \langle D_K(x^{\epsilon_j+\epsilon_{j+r}}):\ 1 \le j \le r\rangle \oplus \langle D(x^{\epsilon_n})\rangle$$
$$N_Q^+ = \langle D_K(x^a):\ a_i = 0\ for\ i \le r,\ i = n, \mid a \mid > 2\rangle$$

One can check, as in the case where L is of type H, that $[N_Q^+, N_Q^+] = 0$ and $[N_Q^-, N_Q^+] \subset N_Q^+$. Moreover, for $1 \le i \le r$, $D_K(x^{3\epsilon_{i+r}})$has weight $(0,...0,\overset{i}{3},0,..0,1)$ and $D_K(x^{4\epsilon_{2r}})$ has weight $(0,...0,4,2)$ which yields $\dim_F T$ linearly independent weights.

We found a subalgebra $Q$ for types $L = W,\ S,\ ,H$ and $K$ which satisfies the assumption in Lemma 2.1 so condition (1) of Theorem 2.2 is satisfied. In each case it is easy to see that $B_Q^- = B_L^-$

so (2) in Theorem 2.2. holds. Now let $V = L_{-1}$ for types $W$, $S$ and $H$. From [S-F] $V$ is the standard module for the reductive Lie algebra $L_0$. Hence $V$ has $\dim_F T$ linearly independent weight vectors having linearly independent weights in $\hat{T}$. For type $K$ $L_0 \cong \mathfrak{sp}(2r) \oplus F$. Again from [S-F] $L_{-1}$ is the standard $\mathfrak{sp}(2r)$ module and $L_{-2}$ is the trivial $\mathfrak{sp}(2r)$ module with $1 \in F$ acting as a non-zero scalar on $L_{-2}$, hence $V = L_{-2} \oplus L_{-1}$ has $\dim_F T$ linearly independent weight vectors having linearly independent weights. Since $V \subset N_Q^- = N_L^-$ (3) of Theorem 2.2 holds and by Corollary 2.4 $U(L)$ has precisely one block.

## 4. Cartan Invariants for Graded Lie Algebras.

Let $L = \oplus_{i=-n}^{m} L_i$ be a restricted graded Lie algebra and set $N^+ = \oplus_{i>0} L_i$, $N^- = \oplus_{i<0} L_i$ and $B^{\pm} = N^{\pm} \oplus L_0$. With this decomposition $U(L)$ satisfies the axioms stated in [Ho-N] and thus a "Brauer-type reciprocity" (described below) must hold.

Let $\{\mathcal{L}(\lambda) : \lambda \in \hat{T}\}$ ($\{L(\lambda) : \lambda \in \hat{T}\}$) denote the set of simple modules of $U(L)$ ($U(L_0)$), and for each $\lambda \in \hat{T}$ let $\mathcal{P}(\lambda)$ ($P(\lambda)$) be the projective indecomposable module with head $\mathcal{L}(\lambda)$ ($L(\lambda)$). Moreover, let s be the number of isomorphism classes of simple $U(L)$ ($U(L_0)$) modules. Set

$$V_{proj}^{\pm}(\lambda) = P(\lambda) \otimes_{U(B^{\pm})} U(L) \text{ and}$$

$$V_{irr}^{\pm}(\lambda) = L(\lambda) \otimes_{U(B^{\pm})} U(L).$$

Moreover, let $DV_{irr}^{\pm}(\mu) = (L(\mu)^* \otimes_{U(B^{\pm})} U(L))^*$ and $DV_{proj}^{\pm} = (P(\mu)^* \otimes_{U(B^{\pm})} U(L))^*$ for all $\mu \in \hat{T}$. ($N^*$ is the contragredient (dual) module of $N$.) It follows from [Ho-N] and [N] that

$$[\mathcal{P}(\lambda) : V_{proj}^{\pm}(\mu)] = [DV_{irr}^{\mp}(\mu) : \mathcal{L}(\lambda)],$$

and if $U(L_0)$ is a symmetric algebra then

$$[\mathcal{P}(\lambda) : V_{irr}^{\pm}(\mu)] = [DV_{proj}^{\mp}(\mu) : \mathcal{L}(\lambda)].$$

The symbols on the left hand side of the equality indicate the number of times $V_{proj}^{\pm}(\mu)$ (resp. $V_{irr}^{\pm}(\mu)$) appears in a *Verma module series* for $\mathcal{P}(\lambda)$. This number was shown to be well-defined in [Ho-N]. The symbols on the right hand side indicate how many times the simple module $\mathcal{L}(\lambda)$ appears in a *Jordan – Holder series* for $D^{\mp}V_{irr}(\mu)$ (resp. $D^{\mp}V_{proj}(\mu)$).

Now assume $L_0$ has a triangular decomposition with respect to a maximal torus $T$ for $L$ and $L_0$: $L_0 = N_{L_0}^- \oplus T \oplus N_{L_0}^+$, and $L_0$ is graded with the torus as zero component. The triangular decomposition for $L_0$ will induce a long triangular decomposition for $L$:

$$L = N_L^- \oplus T \oplus N_L^+$$

by setting $N_L^{\pm} = N^{\pm} \oplus N_{L_0}^{\pm}$ and $B_L^{\pm} = T \oplus N_L^{\pm}$. We will obtain a decompostion of the Cartan matrix for $U(L)$ into a product of matrices involving the Jantzen matrices for $L_0$ and matrices which indicate how the *generalized Verma modules* [Sh] decompose.

Set $c_{\lambda,\mu} = [\mathcal{P}(\lambda) : \mathcal{L}(\mu)]$; then $C = (c_{\lambda,\mu})$ is the *Cartan matrix* for $U(L)$. We need to first show that $\mathcal{P}(\lambda)$ has a filtration with factors isomorphic to Verma modules $V(\mu) = D_\mu \otimes_{U(B_L)} U(L)$. By [Ho-N] $\mathcal{P}(\lambda)$ is filterable in terms of *generalized Verma modules* $V_{proj}^{-} = P(\mu) \otimes_{U(B^-)} U(L)$. From the BGG reciprocity law for classical Lie algebras $P(\mu)$ as $B^-$ module is filterable in terms of the modules

$$W(\sigma) = D_\sigma \otimes_{U(B_{L_0}^-)} U(L_0)$$

with $L_i$ acting trivially for $i < 0$. We claim that

$$W(\sigma) \otimes_{U(B^-)} U(L) \cong D_\sigma \otimes_{U(B_L^-)} U(L).$$

This follows from the fact that for each $\sigma$

$$D_\sigma \otimes_{U(B_L^-)} U(L) \cong (D_\sigma \otimes_{U(B_L^-)} U(B^-)) \otimes_{U(B^-)} U(L)$$

and the observation that

$$W(\sigma) \cong D_\sigma \otimes_{U(B_L)} U(B^-)$$

as $U(B^-)$ modules. Hence $\mathcal{P}(\lambda)$ is filterable in terms of $V(\mu)$ modules. The same argument in [Ho-N] can be used to prove that the $[\mathcal{P}(\lambda) : V(\mu)]$ is well-defined. Now we can write

$$[\mathcal{P}(\lambda) : \mathcal{L}(\mu)] = \sum_{\gamma \in \hat{T}} [\mathcal{P}(\lambda) : D_\gamma \otimes_{U(B_L^-)} U(L)][D_\gamma \otimes_{U(B_L^-)} U(L) : \mathcal{L}(\mu)].$$

If one lets $b_{\lambda,\gamma} = [\mathcal{P}(\lambda) : D_\gamma \otimes_{U(B_L^-)} U(L)]$, $d_{\gamma,\mu} = [D_\gamma \otimes_{U(B_L^-)} U(L) : \mathcal{L}(\mu)]$, $B = (b_{\lambda,\gamma})$ and $D = (d_{\gamma,\mu})$, then $C = BD$. We will first decompose $D$.

Since $L$ has a long triangular decomposition it follows from Theorem 2.2 that

$$[D_\gamma \otimes_{U(B_L^-)} U(L)] = \sum_{\sigma \in \hat{T}} p^\beta [D_\sigma \otimes_{U(B_L^+)} U(L)]$$

where $\beta = dim_F N_L^+ - dim_F N_L^- - dim_F T$. Hence,

$$d_{\gamma,\mu} = \sum_{\sigma \in \hat{T}} p^\beta [D_\sigma \otimes_{U(B_L^+)} U(L) : \mathcal{L}(\mu)].$$

Next observe that

$$\begin{aligned}[D_\sigma \otimes_{U(B_L^+)} U(L) : \mathcal{L}(\mu)] &= [D_\sigma \otimes_{U(B_L^+)} U(B^+) \otimes_{U(B^+)} U(L) : \mathcal{L}(\mu)] \\ &= \sum_{\rho \in \hat{T}} [D_\sigma \otimes_{U(B_L^+)} U(B^+) : L(\rho)][L(\rho) \otimes_{B^+} U(L) : \mathcal{L}(\mu)]\end{aligned}$$

Recall that the simple modules for $U(B^-)$ are obtained from the simple modules for $U(L_0)$ by letting $N^-$ act trivially. Therefore, for all $\sigma, \rho \in \hat{T}$ we have

$$\begin{aligned}[D_\sigma \otimes_{U(B_L^+)} U(B^+) : L(\rho)] &= [(D_\sigma \otimes_{U(B_L^+)} U(B^+)) \mid_{L_0} : L(\sigma)] \\ &= [D_\sigma \otimes_{U(B_{L_0}^+)} U(L_0) : L(\sigma)].\end{aligned}$$

Now let $j_{\sigma,\rho} = [D_\sigma \otimes_{U(B_{L_0}^+)} U(L_0) : L(\rho)]$, $k_{\rho,\mu} = [L(\rho) \otimes_U (B^+) U(L) : \mathcal{L}(\mu)]$, $J = (j_{\sigma,\rho})$, $K = (k_{\rho,\mu})$ and $A = (p^\beta)$ where $\beta = p^{dim_F N_L^+ - dim_F N_L^- - dim_F T}$. Then $D = AJK$.

Next we will decompose $B$. The reciprocity law hold for the $V(\mu)$ by using the same arguments in [Ho-N], so it follows that

$$\begin{aligned} b_{\lambda,\gamma} &= [\mathcal{P}(\lambda) : D_\gamma \otimes_{U(B_L^-)} U(L)] \\ &= [(D_\gamma^* \otimes_{U(B_L^+)} U(L))^* : \mathcal{L}(\lambda)] \quad (\dagger) \\ &= [D_\gamma^* \otimes_{U(B_L^+)} U(L) : \mathcal{L}(\lambda)^*].\end{aligned}$$

By applying the same arguments used in finding the decomposition of $D$ we have

$$\begin{aligned}[D_\gamma^* \otimes_{U(B_L^+)} : \mathcal{L}(\lambda)^*] &= \sum_{\alpha \in \hat{T}} [D_\gamma^* \otimes_{U(B_L^+)} U(B^+) : L(\alpha)][L(\alpha) \otimes_{U(B^+)} U(L) : \mathcal{L}(\lambda)^*] \\ &= \sum_{\alpha \in \hat{T}} [D_\gamma^* \otimes_{U(B_{L_0}^+)} U(L_0) : L(\alpha)][L(\alpha) \otimes_{U(B^+)} U(L) : \mathcal{L}(\lambda)^*].\end{aligned}$$

Set $J' = ([D_\gamma^* \otimes_{U(B_{L_0}^+)} U(L_0)])$ and $K' = ([L(\alpha) \otimes_{U(B^+)} U(L) : \mathcal{L}(\lambda)^*])$. Then $B = K'^T J'^T$. In summary we state:

**Theorem 4.1.** *Let $L$ be a restricted Lie algebra satisfying the assumptions in this section. Then the Cartan matrix for $U(L)$ can be written a product of sxs matrices:*

$$C = K'^T J'^T AJK$$

*where*

$$J = ([D_\alpha \otimes_{U(B_{L_0}^+)} U(L_0) : L(\tau)]), \quad J' = ([D_\gamma^* \otimes_{U(B_{L_0}^+)} U(L_0) : L(\tau)]),$$

$$K = ([L(\tau) \otimes_{U(B^+)} U(L) : \mathcal{L}(\lambda)]), \quad K' = ([L(\sigma) \otimes_{U(B^+)} U(L) : \mathcal{L}(\mu)^*]) \textit{ and}$$
$$A = (a) \quad \textit{where} \quad a = p^{dim_F N_L^+ - dim_F N_L^- - dim_F T}.$$

The decomposition of $C$ in Theorem 4.1 takes a simpler form if $L$ safisfies the following additional assumption.

**Hypothesis 4.2.** *There exists a one to one correspondence $\phi : \hat{T} \longrightarrow \hat{T}$, with*
$$[(D^*_\gamma \otimes_{U(B_L^+)} U(L))^*] = [D_{\phi(\gamma)} \otimes_{U(B_L^+)} U(L)].$$

By replacing $(D^*_\gamma \otimes_{U(B_L^+)} U(L))^*$ by $D_{\phi(\gamma)} \otimes_{U(B_L^+)} U(L)$ in (†) we can write $B = KJ'^T$ where $J' = ([D_{\phi(\gamma)} \otimes_{U(B_{L_0}^+)} U(L_0) : L(\alpha)])$. Moreover, the matrix $A$ has the same number in each entry so permuting the columns of $J'^T$ does not change the product $J'^T A$. Therefore, if $L$ satisfies Hypothesis 4.2 then the following theorem holds.

**Theorem 4.3.** *Let L be a restricted Lie algebra the assumptions in this section along with Hypothesis 4.2. Then the Cartan matrix for $U(L)$ can be written as a product of matrices*
$$C = (JK)^T A(JK)$$
*where*
$$J = ([D_\alpha \otimes_{U(B_{L_0}^+)} U(L_0) : L(\tau)]) \quad K = ([L(\tau) \otimes_{U(B^+)} U(L) : \mathcal{L}(\lambda)]),$$
$$A = (a) \quad \textit{with} \quad a = p^{dim_F N_L^+ - dim_F N_L^- - dim_F T}.$$
*In particular the Cartan matrix is symmetric.*

The Lie algebras of Cartan type satisfy the conditions stated in this section. For $L = W(m,1)$, $S(m,1)$ or $H(2m,1)$ (resp.) there is a grading such that $L = L_{-1} \oplus L_0 \oplus L_1 \oplus ...L_n$ where $L_0$ is isomorphic to $\mathfrak{g}l(m)$, $\mathfrak{s}l(m)$ or $\mathfrak{s}p(2m)$ (resp.). In the case where $L = K(m,1)$, $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus ...L_n$ where $L_0 = \mathfrak{s}p(2m) \oplus F$. With all Lie algebras of Cartan type the $L_0$ has a triangular decomposition relative to a maximal torus for both $L_0$ and $L$. Furthermore, $L_0$ is either classical or reductive so $L_0$ has a grading with the maximal torus as the zero component [Ho-N]. We saw in section 3 that the negative and positive parts of this triangular decomposition can be coupled with the negative and positive parts of the grading to obtain a long triangular decomposition for $L$ which satisfy the assumptions in this section.

In [Sh] Shen classifies all restricted irreducible modules for the Lie algebras W(m,1), S(m,1) and H(m,1) and decomposes the generalized Verma modules $L(\tau) \otimes_{U(B^+)} U(L)$. This in turn allows us to state the following theorems, proven for W(m,1) in [N], which generalize to S(m,1) and H(m,1).

**Theorem 4.4.** *For $L = W(m,1),\ S(m,1)$ and $H(m,1)$ there exists a one to one correspondence $\phi : \hat{T} \longrightarrow \hat{T}$ such that*

$$[(L(\mu) \otimes_{U(B^+)} U(L))^*] = [L(\phi(\mu)) \otimes_{U(B^+)} U(L)].$$

By using Theorem 4.4. and the fact that in the classical $L_0$ component $\phi : \hat{T} \longrightarrow \hat{T}$ preserves linkage classes we obtain:

**Corollary 4.5.** *If $L = W(m,1),\ S(m,1)$ and $H(m,1)$ then there exists a one to one correspondence $\phi : \hat{T} \longrightarrow \hat{T}$ between Verma modules such that*

$$[(D_\mu \otimes_{U(B_L^+)} U(L))^*] = [D_{\phi(\mu)} \otimes_{U(B_L^+)} U(L)].$$

Therefore, $W(m,1),\ S(m,1)$, and $H(m,1)$ satisfy Hypothesis 4.2 and we can apply Theorem 4.3. to get a decomposition of the Cartan matrix $C = (JK)^T A(JK)$. For the classical Lie algebra $sl(2)$ Pollack [Po1] [Po2] computed the Cartan invariants. In [N] the Cartan matrix was computed for the Cartan type Lie algebras $L = W(1,n)$ and $W(2,n)$. Next we will show how to compute Cartan invariants for $L = H(2,1)$. In doing so we will complete the explicit computation of Cartan invariants and the dimension of the projective covers for the restricted universal enveloping algebras associated with the restricted simple toral rank one Lie algebras.

Let $L = H(2,1)$ be the Hamiltonian algebra of dimension $p^2 - 2$. From section 3 we know that $L$ has a grading $L_{-1} \oplus L_0 \oplus L_1 \oplus ...L_n$ such that $L_0 \cong \mathfrak{sp}(2) \cong \mathfrak{sl}(2)$. Pollack [Po1] calculated the multiplicities of simple modules in the Verma modules for $\mathfrak{sl}(2)$. From his results we can write

$$J = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & \dots & \dots & 0 \\ 1 & 1 & 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & 0 & 1 & 1 & \dots & \dots & \dots & 0 \\ 0 & 0 & 1 & 1 & \dots & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & 1 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 1 & 1 & 0 \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 & 1 \end{pmatrix}$$

with rows and columns parmetrized by the set of weights

$$0,\ p-2,\ 1,\ p-3, ...\frac{p-1}{2},\ \frac{p-3}{2},\ p-1.$$

Shen [Sh] decomposed the generalized Verma modules $L(\lambda)\otimes_{U(B^+)} U(L)$ for $L = H(2,1)$. As elements in the Grothendick ring we have

$$\begin{aligned}
&[L(\lambda)\otimes_{U(B^+)} U(L)] = [\mathcal{L}(\lambda)] \quad for \quad \lambda \neq 0,1\\
&[L(0)\otimes_{U(B^+)} U(L)] = 2[\mathcal{L}(0)] + [\mathcal{L}(1)]\\
&[L(1)\otimes_{U(B^+)} U(L)] = 4[\mathcal{L}(0)] + 2[\mathcal{L}(1)]
\end{aligned}$$

with

$$\begin{aligned}
&dim_F\mathcal{L}(\lambda) = (\lambda+1)p^2 \quad \lambda \neq 0,1\\
&dim_F\mathcal{L}(0) = 1 \quad dim_F\mathcal{L}(1) = p^2-2.
\end{aligned}$$

Therefore,

$$K = \begin{pmatrix}
2 & 0 & 1 & 0 & 0 & \dots & \dots & 0\\
0 & 1 & 0 & 0 & 0 & \dots & \dots & 0\\
4 & 0 & 2 & 0 & 0 & \dots & \dots & 0\\
0 & 0 & 0 & 1 & 0 & \dots & \dots & 0\\
0 & 0 & 0 & 0 & 1 & 0 & \dots & 0\\
0 & \dots & \dots & \dots & \dots & \ddots & \dots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & \dots & \dots & \dots & \dots & \dots & \dots & 1
\end{pmatrix}$$

with rows and columns parametrized in the same manner as $J$. The matrix $A$ has all entries equal to $p^{p^2-10}$. Hence,

$$C = (JK)^T A(JK) = p^{p^2-10}\begin{pmatrix}
144 & 24 & 72 & 24 & \dots & 24 & 12\\
24 & 4 & 12 & 4 & \dots & 4 & 2\\
72 & 12 & 36 & 12 & \dots & 12 & 6\\
24 & 4 & 12 & 4 & \dots & 4 & 2\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
24 & 4 & 12 & 4 & \dots & 4 & 2\\
12 & 2 & 6 & 2 & \dots & 2 & 1
\end{pmatrix}$$

**Theorem 4.6.** *The projective indecomposable modules for $U(L)$ where $L = H(2,1)$ have the following dimensions*

$$\begin{aligned}
&dim_F\mathcal{P}(0) = 12p^{p^2-6}\\
&dim_F\mathcal{P}(1) = 6p^{p^2-6}\\
&dim_F\mathcal{P}(p-1) = p^{p^2-6}\\
&dim_F\mathcal{P}(\lambda) = 2p^{p^2-6} \quad for \quad \lambda \neq 0,1,p-1.
\end{aligned}$$

*Moreover, these projective modules can be expressed in terms of sums of Grothendick ring elements as follows.*

$$[\mathcal{P}(0)] = 144p^{p^2-10}[\mathcal{L}(0)] + 72p^{p^2-10}[\mathcal{L}(1)] + \sum_{j=2}^{p-2} 24p^{p^2-10}[\mathcal{L}(j)] + 12p^{p^2-10}[\mathcal{L}(p-1)]$$

$$[\mathcal{P}(1)] = 72p^{p^2-10}[\mathcal{L}(0)] + 36p^{p^2-10}[\mathcal{L}(1)] + \sum_{j=2}^{p-2} 12p^{p^2-10}[\mathcal{L}(j)] + 6p^{p^2-10}[\mathcal{L}(p-1)]$$

$$[\mathcal{P}(p-1)] = 12p^{p^2-10}[\mathcal{L}(0)] + 6p^{p^2-10}[\mathcal{L}(1)] + \sum_{j=2}^{p-2} 2p^{p^2-10}[\mathcal{L}(j)] + p^{p^2-10}[\mathcal{L}(p-1)]$$

$$[\mathcal{P}(\lambda)] = 24p^{p^2-10}[\mathcal{L}(0)] + 12p^{p^2-10}[\mathcal{L}(1)] + \sum_{j=2}^{p-2} 4p^{p^2-10}[\mathcal{L}(j)] + 2p^{p^2-10}[\mathcal{L}(p-1)]$$

$$for\ \lambda \neq 0, 1, p-1$$

In conclusion we would like to state a conjecture which may lead to further study in connection with the structure and representation theory of restricted Lie algebras.

**Conjecture.** *If $L$ is a strongly degenerate restricted Lie algebra and $L$ contains no classical or toral ideals then $U(L)$ has precisely one block.*

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